Journal of Pure and Applied Mathematics: Advances and Applications
Volume 4, Number 1, 2010, Pages 23-37

SEI EPIDEMIC MODELS WITH NON-LINEAR INCIDENCE RATE AND UNIFORM REPRODUCTION

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Abstract

In this paper, a general SEI epidemic model is considered with nonlinear incidence rate and uniform reproduction rate more general than those in the literature. Disease-free and endemic points are discussed with their stabilities. Hopf bifurcation and periodic solution are studied. Numerical discussion is given to show the effects of changing parameters of the system.

1. Introduction

In this paper, we are mainly concerned with a generalized SEI epidemic system of the form

2010 Mathematics Subject Classification: 74H55.

Keywords and phrases: equilibria, stability, Hopf bifurcation.

Received July 5, 2010

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\begin{aligned}
\frac{dX}{dt} &= B(N)(X + \nu_1 W + \rho\nu_2 Y) - Nf(\frac{X}{N}, \frac{Y}{N}, N) - D(N)X, \\
\frac{dW}{dt} &= Nf(\frac{X}{N}, \frac{Y}{N}, N) - \varepsilon W - D(N)W, \qquad (1) \\
\frac{dY}{dt} &= B(N)(1 - \rho)\nu_2 Y - [\alpha + D(N)]Y, \\
\frac{dN}{dt} &= B(N)(X + \nu_1 W + \nu_2 Y) - \alpha Y - D(N)N.
\end{aligned}
$$

Where $X(t)$, $W(t)$, and $Y(t)$ are the number of individuals, who are susceptible, exposed, and infectious, respectively, in the population of size $N(t)$ and $f(\frac{X}{N}, \frac{Y}{N}, N)$ represent a generalized incidence. The number of new cases per unit time. $B(N)(X + \nu_1 W + \nu_2 Y)$ is the rate of newborns in population, αY is the disease-related death rate, $B(N)$ $(1 - \rho)\nu_2 Y$ expresses the flow of vertically infected new borns into the infectious class, ρ is the fraction not infected, and ν_1 and ν_2 represent reduced reproduction parameters. In this paper, we consider an exponential demographic structure at which $B(N) = b$ and $D(N) = d$. Since the population size grows exponentially in the case, who the net growth rate $r = b - d$ is positive. Here b is the birth rate, d is the death rate, α is the rate that the infective population become recovered, and ε is the rate that the exposed population become infective.

We suppose that the non-linear incidence $f(S, I, N)$ satisfies the following conditions for all $S, I, N \geq 0$:

(c_1) $f(S, I, N) \geq 0$;

(c_2) $\frac{\partial f}{\partial S}, \frac{\partial f}{\partial I}, \frac{\partial f}{\partial N} \geq 0$;

(c_3) $f(0, I, N) = f(S, 0, N) = f(S, I, 0) = 0$;

(c_4) ε, b, d, r are positive and α is nonnegative parameter.

2. Main Results

We first consider the SEI model given in system (1) with uniform reproduction $(\nu_1 = \nu_2 = 1)$. This model has no vertical transmission, (i.e., $\rho = 1$) in terms of $S(t) = \frac{X(t)}{N(t)}$ and $I(t) = \frac{Y(t)}{N(t)}$. This leads to system

$$\frac{dS}{dt} = b(1 - S) - f(S, I, N) + \alpha IS;$$

$$\frac{dI}{dt} = \varepsilon(1 - S) - (\varepsilon + \alpha + b - \alpha I)I; \tag{2}$$

$$\frac{dN}{dt} = (r - \alpha I)N.$$

Theorem 1. *Let U be the region defined by $U = \{(S, I, N) \in R_+^3, S, I \geq 0, S + I \leq 1, 0 \leq N < \infty\}$, then we have the following result occurs with respect to system* (2):

(i) *U is positively invariant.*

(ii) *All solutions of the system* (2) *with initial values in R_+^3 are eventually uniformly bounded and attracted into the region.*

(iii) *System* (2) *is dissipative.*

Proof. Since by the third equation of (2), we have

$$N = N_\circ \exp \int_{t_\circ}^{t} (r - \alpha I) ds,$$

then $0 \leq N < \infty$.

Moreover by (2), we have

$$\frac{d(S + I)}{dt} = (b + \varepsilon)(1 - S) - f(S, I, N) + \alpha IS - (\varepsilon + \alpha + b - \alpha I)I.$$

In view of (c_1), we have

$$\frac{d(S+I)}{dt} \leq (b+\varepsilon)(1-S) - (\varepsilon+b)I - \alpha I + \alpha I^2 + \alpha IS$$

$$< (b+\varepsilon)(1-S) - (\varepsilon+b)I - \alpha IE.$$

But since $E = 1 - S - I$ and $E > 0$, it follows that

$$\frac{d(S+I)}{dt} < (b+\varepsilon)(1-(S+I)).$$

Thus $(S+I) < 1 - (1 - (S_\circ + I_\circ))\exp-(\varepsilon+b)(t-t_\circ)$.

Thus, $\limsup_{t\to\infty}(S+I) \leq 1$, i.e., the solutions of system (2) are uniformly bounded, which implies that U is positively invariant and the dissipativity of system (2) is proved (see [6]).

The region U in the SIN space system (2) always has the disease-free equilibrium $P_\circ \equiv (1, 0, 0)$ and has an interior equilibrium $P_1 = (1-\psi, \frac{r}{\alpha}, N_2)$, where $f(1-\psi, \frac{r}{\alpha}, N_1) = r + d\psi$ and $\psi = (\frac{r}{\alpha})(\frac{\varepsilon+\alpha+d}{\varepsilon})$ (see [4]).

Theorem 2. *The disease-free equilibrium* $P_\circ$ *of* (2) *is unstable.*

Proof. Since the Jacobian of the system (2) is

$$\mathrm{J} = \begin{bmatrix} -b - \frac{\partial f}{\partial S} + \alpha I & \alpha S - \frac{\partial f}{\partial I} & -\frac{\partial f}{\partial N} \\ -\varepsilon & -(\varepsilon+\alpha+b-2\alpha I) & 0 \\ 0 & -\alpha N & r - \alpha I \end{bmatrix},$$

the characteristic equation at $P_\circ = (1, 0, 0)$ is

$$(r-\lambda)\Big[\lambda^2 + \lambda\Big(\frac{\partial f(1,0,0)}{\partial S} + \varepsilon + \alpha + 2b\Big) + (\varepsilon+\alpha+b)\Big(b + \frac{\partial f(1,0,0)}{\partial S}\Big)$$

$$+ \Big(\alpha - \frac{\partial f(1,0,0)}{\partial I}\Big)\varepsilon\Big] = 0.$$

The characteristic roots are

$$\lambda_1 = r \text{ and } \lambda_{2,3} = \frac{-\left(\frac{\partial f(1,0,0)}{\partial S} + \varepsilon + \alpha + 2b\right) \pm \sqrt{\left(\varepsilon + \alpha - \frac{\partial f(1,0,0)}{\partial S}\right)^2 - 4\alpha\varepsilon + 4\frac{\partial f(1,0,0)}{\partial I}\varepsilon}}{2}.$$

Since $r > 0$, then $P_\circ$ is unstable. This completes the proof.

Theorem 3. *All solution paths in U on the $N = 0$ plane approach the equilibrium $P_\circ(1, 0, 0)$.*

Proof. Consider the Lyapunov function

$$V = \varepsilon E + (b + \varepsilon)I = \varepsilon - \varepsilon S + bI;$$

$$\frac{dV}{dt} = -\varepsilon\frac{dS}{dt} + b\frac{dI}{dt}$$

$$= -\varepsilon\alpha IS - b(\varepsilon + \alpha + b - \alpha I)I + f(S, I, N).$$

In the $N = 0$ plane, $\frac{dV}{dt} = -\varepsilon\alpha IS - b(\varepsilon + \alpha + b - \alpha I)I \leq 0.$

The largest positively invariant set of the subset, where $\frac{dV}{dt} = 0$ is the equilibrium $P_\circ(1, 0, 0)$, so that all paths in the $N = 0$ plane approach P_1 by the La salle theorem [17].

The following theorem gives sufficient conditions for the existence of periodic solutions.

Theorem 4. *The system* (2) *has a periodic solution in the neighborhood of P_1, if*

(i) $\varepsilon + \alpha + 2d + \dfrac{\partial f(1 - \psi, \frac{r}{\alpha}, N_1)}{\partial S} > r,$

(ii) $(\varepsilon + \alpha + 2d - r + A_1)[\varepsilon\alpha(1 - \psi) + (\varepsilon + \alpha + d - r)(d + A_1) - \varepsilon A_2]$

$= \alpha\varepsilon N_1 A_3,$

where

$$A_1 = \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial S}, \; A_2 = \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial I}, \; A_3 = \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial N}.$$

Proof. For the endemic point $P_1 = (1-\psi, \frac{r}{\alpha}, N_1)$, the characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = \varepsilon + \alpha + 2d - r + \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial S},$$

$$a_2 = \varepsilon\left(\alpha(1-\psi) - \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial I}\right) + (\varepsilon + \alpha + d - r)\left(d + \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial S}\right),$$

$$a_3 = \alpha\varepsilon N_1 \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial N}.$$

Since by (i), we have $a_1 > 0$ and $a_3 > 0$.

Hence Hopf bifurcation occurs at the neutral stability surface $a_1a_2 - a_3 = 0$, see [14].

This leads to $(\varepsilon + \alpha + 2d - r + A_1)[\varepsilon\alpha(1-\psi) + (\varepsilon + \alpha + d - r)(d + A_1) - \varepsilon A_2] = \alpha\varepsilon N_2 A_3$,

where

$$\frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial S} = A_1, \quad \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial I} = A_2, \quad \frac{\partial f(1-\psi, \frac{r}{\alpha}, N_1)}{\partial N} = A_3.$$

Thus under (i) and (ii), we conclude that system (2) has a periodic solution in the neighborhood of P_1.

3. A Special Case

In this section, we consider the model (2) in the special form of $f(S, I, N) = \beta NH(I, S)$.

Assume that the nonlinear function $H(S, I)$ satisfies the following conditions for all $S, I \geq 0$:

(c_5) $H(S, I) \geq 0$;

(c_6) $\frac{\partial H}{\partial S} \geq 0$, and $\frac{\partial H}{\partial I} \geq 0$;

(c_7) $H(0, I) = H(S, 0) = 0$.

The model (2) takes the form

$$\begin{aligned} \frac{dS}{dt} &= b(1 - S) - \beta NH(I, S) + \alpha IS; \\ \frac{dI}{dt} &= \varepsilon(1 - S) - (\varepsilon + \alpha + b - \alpha I)I; \\ \frac{dN}{dt} &= (r - \alpha I)N. \end{aligned} \tag{3}$$

Theorem 5. *Let U be a region defined as in Theorem* 1, *then*

(i) *U is positively invariant.*

(ii) *All solutions of system* (3) *with initial values in* R_+^3 *are eventually uniformly bounded and attracted into the region U.*

(iii) *The system* (3) *is dissipative.*

The region U in SIN space system (3) always has a disease-free equilibrium $P_\circ = (1, 0, 0)$ and $P_1 = (1 - \psi, \frac{r}{\alpha}, \frac{d\psi + r}{\beta H(1 - \psi, \frac{r}{\alpha})})$. Note that as $\psi \to 1$, the value $H(1 - \psi, \frac{r}{\alpha}) = 0$ implies that P_1 approaches the pseudo equilibrium $P_2 = (0, \frac{r}{\alpha}, \infty)$.

The following theorem shows that the disease-free equilibrium $P_\circ$ is unstable.

Theorem 6. *The disease-free equilibrium* $P_\circ$ *of* (3) *is unstable.*

Proof. Since the Jacobian of the system (3) is

$$\mathbf{J} = \begin{bmatrix} -b - N\beta\dfrac{\partial H}{\partial S} + \alpha I & -N\beta\dfrac{\partial H}{\partial I} + \alpha S & -\beta H(S, I) \\ -\varepsilon & -(\varepsilon + \alpha + b) + 2\alpha I & 0 \\ 0 & -\alpha N & (r - \alpha I) \end{bmatrix},$$

then the characteristic equation at $P_\circ(1, 0, 0)$ is

$$(r - \lambda)((b + \alpha) + \lambda)(\varepsilon + b + \lambda) = 0.$$

The eigenvalues are $r > 0, -(b + \alpha),$ and $-(\varepsilon + b)$.

Then $P_\circ(1, 0, 0)$ is unstable saddle point.

Theorem 7. *All solution paths in U on the* $N = 0$ *plane approach the equilibrium* $P_\circ(1, 0, 0)$.

Proof. The proof is the same as in Theorem 3.

Theorem 8. *The system* (3) *has a periodic solution in the neighborhood of* P_1, *if*

(i) $\varepsilon + \alpha + 2d + (\dfrac{d\psi + r}{A_3})A_1 > r$;

(ii) $(\varepsilon + \alpha + 2d - r + (\dfrac{d\psi + r}{A_3})A_1)[\varepsilon\alpha(1 - \psi) + (\varepsilon + \alpha + d - r)$

$$\times (d + (\frac{d\psi + r}{A_3})A_1) - \varepsilon(\frac{d\psi + r}{A_3})A_2] = (d\psi + r)\alpha\varepsilon;$$

where $A_1 = \dfrac{\partial H(S_1, I_1)}{\partial S}$, $A_2 = \dfrac{\partial H(S_1, I_1)}{\partial I}$, $A_3 = H(S_1, I_1)$.

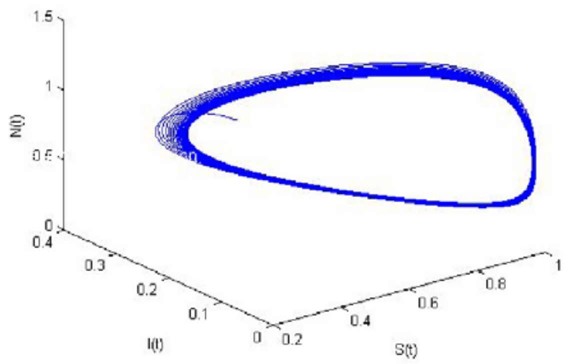
Proof. The proof is similar to the proof of Theorem 4.

4. Numerical Example

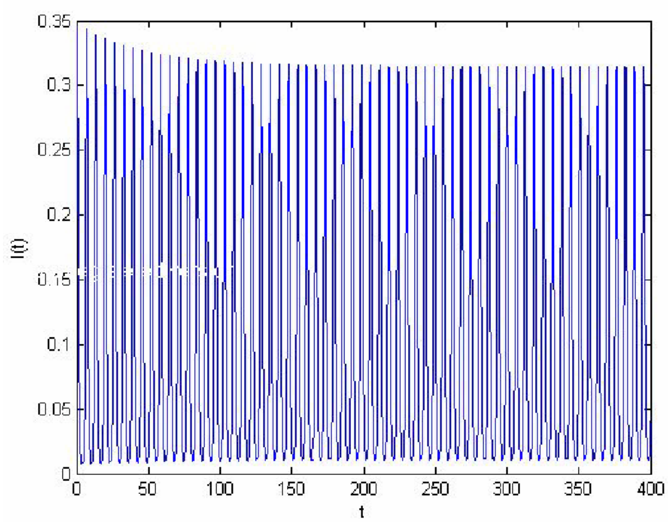
In this section, we consider a special form of the above systems and study numerically their behavior according to the values of parameters. First, we consider $f(S, I, N) = \beta NH(I, S) = \beta NSI$. So, the systems take the form

$$\begin{aligned} \frac{dS}{dt} &= b(1-S) - \beta SIN + \alpha IS; \\ \frac{dI}{dt} &= \varepsilon(1-S) - (\varepsilon + \alpha + b - \alpha I)I; \\ \frac{dN}{dt} &= (r - \alpha I)N; \end{aligned} \tag{4}$$

(where βXY is called *the simple mass action incidence*). Using fourth order Runge-Kutta method and consider the parameter values $\varepsilon = 5$, $\beta = 10$, $b = 0.55$, $d = 0.05$, and $\alpha = 5$, (these values are consistent with those used in [4]). The projection of the solution of Equation (4) in R^3 space and time response of $I(t)$, when the initial conditions are taken to be $S_\circ = 0.4$, $I_\circ = 0.2$, and $N_\circ = 1$ (see Figure 1).



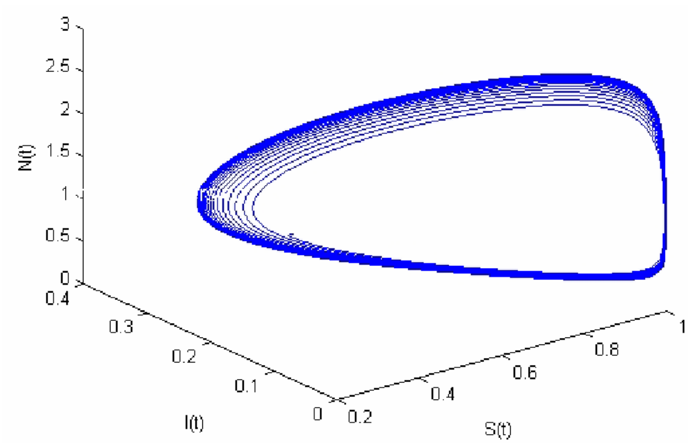
(a) The projection of the solution of Equation (4) in R^3.



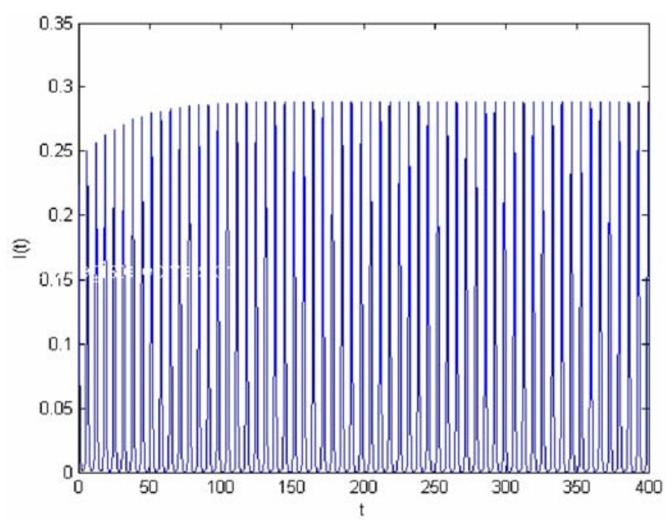
(b) Time response of $I(t)$.

Figure 1

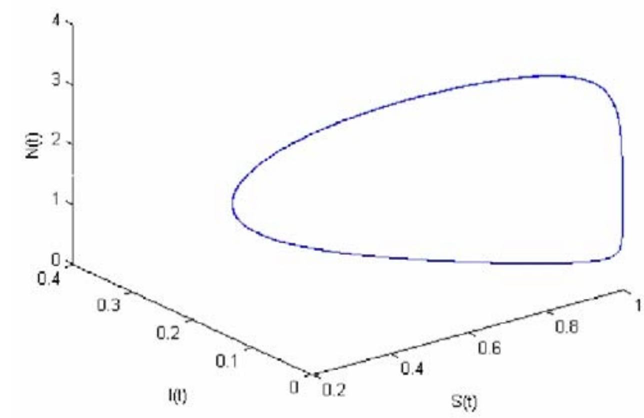
To study the effect of parameter α, we fixed the other parameters we found that the projection of solution has three different cases, that is, (a) chaotic attractor, (c) limit cyclic, and (e) spiral focus. Figure 2 gives the projection of solution of Equation (4) in R^3 space and time response of I, when the initial conditions are taken to be $S_\circ = 0.4$, $I_\circ = 0.2$, and $N_\circ = 1$ (see Figure 2). Let $\varepsilon = 5$, $\beta = 10$, $b = 0.55$, $d = 0.05$, and $\alpha = 9$, 12.2, 3.3.



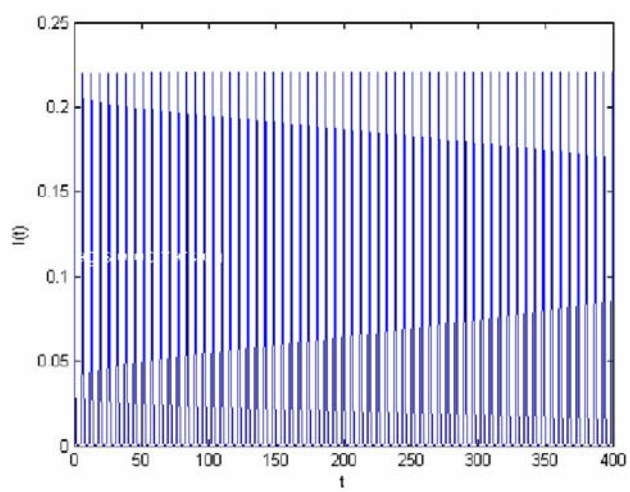
(a) Chaotic attractor for $\alpha = 9$.



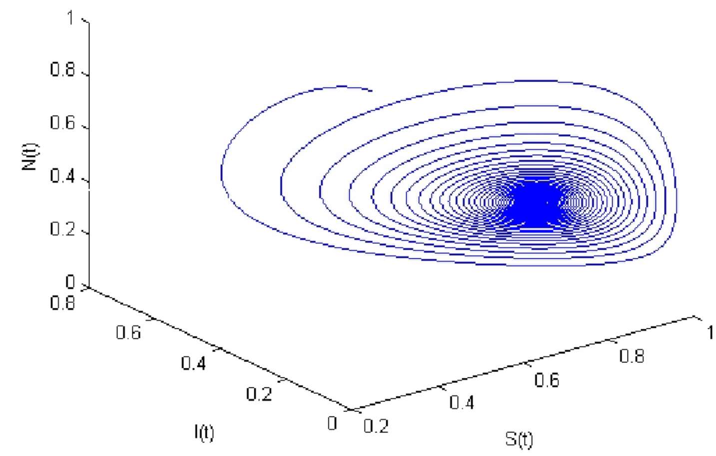
(b) Time response of $I(t)$ for $\alpha = 9$.



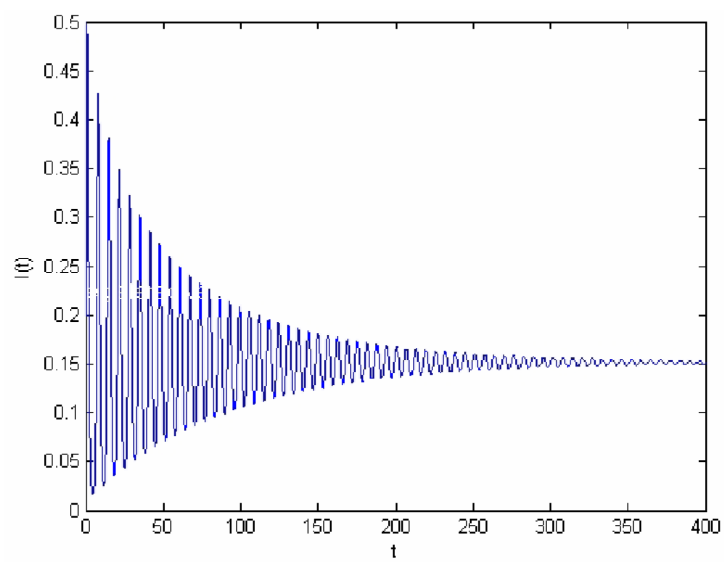
(c) Limit cyclic for $\alpha = 12.2$.



(d) Time response of $I(t)$ for $\alpha = 12.2$.



(e) Spiral focus for $\alpha = 3.3$.



(f) Time response of $I(t)$ for $\alpha = 3.3$.

Figure 2

We can also get different cases similar to previous cases as ε, b, and d are parameters, but with different value.

5. Conclusion

In this paper, we discussed a general SEI model and we introduced a special form numerically by using fourth order Runge-Kutta method through Matlab program. The parameters affect on the solution behavior, for some values of parameters are found. We get asymptotically stable solution (Figure 2(e)), other value we have periodic solution (Figure 2(c)), and chaotic attractor (Figure 2(a)). The obtained results are consistent with those in [5].

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